Semi Nilpotent Elements

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Abstract: In this paper we study semi nilpotent elements in rings. It is shown that every element of \( \mathbb{Z}_n \) where \( n \) is square free is a trivial semi nilpotent. It is proved that every nontrivial nilpotent element is a nontrivial semi nilpotent. Conditions are given under which every element of the group ring \( \mathbb{Z}_nG \) is semi nilpotent. It is shown that if \( p \) is prime and \( p \) divides the order of \( G \), then \( \mathbb{Z}_pG \) has nontrivial semi nilpotent. Also it is proved that if \( G \) is a cyclic group of order \( q^n \), then every element of \( \mathbb{Z}_pG \), \( p \) is prime, is semi nilpotent.

Keywords: Nilpotent, Semi Nilpotent

1. Introduction

The concept of semi nilpotent element was introduced by Vasantha Kandasamy (1997). An element \( x \) of a ring \( R \) is semi nilpotent if \( x^n - x \) is a nilpotent element of \( R \), for some positive integer \( n > 1 \). If \( x^n - x = 0 \), then \( x \) is said to be a trivial semi nilpotent. Clearly every nilpotent element is semi nilpotent and every idempotent is semi nilpotent. In Kandasamy (2002), it is shown that if \( K \) is a field of characteristic 0 and \( G \) is a torsion free abelian group, then the group ring \( KG \) has no nontrivial semi nilpotent element. In this work by using some well-known theorems in number theory the form of both trivial and nontrivial semi nilpotent elements in \( \mathbb{Z}_n \) are given. It is shown that every element except 0 and 1 in the group ring \( \mathbb{Z}_2G \) where \( G \) is a cyclic group of order \( 2^n \), for even \( n \), is a nontrivial semi nilpotent element. At the end we answer an open problem given in Kandasamy (2002) concerning such elements, Theorem 2.10 and Theorem 2.12.

2. Semi Nilpotents

In this section, we study semi nilpotent elements in \( \mathbb{Z}_n \), and in the group ring \( \mathbb{Z}_nG \) for a cyclic group \( G \) of finite order.

**Proposition 2.1.** Every nontrivial nilpotent element in \( \mathbb{Z}_n \), with the prime factorization of \( n = p_1^{a_1}p_2^{a_2}...p_k^{a_k} \) is a nontrivial semi nilpotent.

**Proof.** Let \( 0 \neq x \) be a nilpotent element of \( \mathbb{Z}_n \) and \( m \) be the least positive integer such that \( x^m \equiv 0 \pmod{n} \). Then \( x = p_1^{\ell_1}p_2^{\ell_2}...p_k^{\ell_k} \), such that at least one of \( \ell_i \leq a_i \) say \( \ell_1 < a_1 \). We have to show that \( x^t - x \not\equiv 0 \pmod{n} \) for each positive integer \( t \). Suppose \( x^t - x \equiv 0 \pmod{n} \) for some
$t > 1$, so $x \left(x^{t-1} - 1\right) \equiv 0 \pmod{n}$, which means that $n \mid x \left(x^{t-1} - 1\right)$, which implies $p_1\alpha_1 \mid x \left(x^{t-1} - 1\right)$. But $t \leq \alpha_1$, so $p_1\alpha_1 \nmid x$ and clearly $x^{t-1} - 1$ is not divisible by $p_1$, so $p_1\alpha_1 \nmid x \left(x^{t-1} - 1\right)$, contradiction. 

**Proposition 2.2.** Every element in $\mathbb{Z}_n$, where $n$ is a square free is a trivial semi nilpotent element.

**Proof.** Since $n$ is square free, then $n = p_1 p_2 \ldots p_k$ (Dummit & Foote, 2004), for some distinct primes $p_i$, $1 \leq i \leq k$. Now let $a \in \mathbb{Z}_n$. If $p_i \nmid a$, for each $1 \leq i \leq k$, then by Euler’s Theorem (Burton, 1980), $a^{\phi(p_1p_2\ldots p_k)} \equiv 1 \pmod{n}$, that is $a^{(p_1-1)(p_2-1)\ldots(p_k-1)} \equiv 1 \pmod{n}$, which means

$$a^{(p_1-1)(p_2-1)\ldots(p_k-1)+1} \equiv a \pmod{n},$$

so $a$ is a trivial semi nilpotent element in $\mathbb{Z}_n$. Now, suppose $a$ is divisible by some of $p_i$’s, without loss of generality, suppose $a$ is divisible by $p_1$. For each $1 \leq i \leq k$, and $a$ is not divisible by $p_{i+1}, p_{i+2}, \ldots, p_k$. Hence $a = t_1 p_1 = t_2 p_2 = \ldots = t_\ell p_\ell$ for some $t_i \in \mathbb{Z}^+$ with $p_j \nmid t_i$, $1 \leq j \leq \ell$ and by Fermat’s Little Theorem (Schroeder, 2006),

$$(t_1 p_1)(p_{\ell+1}-1)(p_{\ell+2}-1)\ldots(p_k-1) \equiv 1 \pmod{p_{\ell+1}p_{\ell+2}\ldots p_k}.$$

So,

$$t_1 p_1 p_{\ell+1} p_{\ell+2} \ldots p_k \mid t_1 p_1 a^{(p_{\ell+1}-1)(p_{\ell+2}-1)\ldots(p_k-1)} - t_1 p_1.$$

But $p_i \nmid t_i p_1$ for each $2 \leq i \leq \ell$, then

$$p_1 p_2 \ldots p_k \mid t_1 p_1 a^{(p_{\ell+1}-1)(p_{\ell+2}-1)\ldots(p_k-1)} - t_1 p_1,$$

this means $a^{(p_{\ell+1}-1)(p_{\ell+2}-1)\ldots(p_k-1)+1} \equiv a \pmod{n}$.

Hence $a$ is a trivial semi nilpotent element. 

**Theorem 2.3.** Consider $\mathbb{Z}_n$, with the prime factorization of $n = p_1 t_1, p_2 t_2, \ldots, p_\ell t_\ell$, with at least one of $t_i > 1$ and let $0 \neq \omega \in \mathbb{Z}_n$. Then $\omega$ is a trivial semi nilpotent if $\omega$ has the form $\omega = q_1^{s_1} q_2^{s_2} \ldots q_\ell^{s_\ell} r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_\ell^{\alpha_\ell}$, with $\ell < k$, $\alpha_i \geq 0$, and for each $1 \leq i \leq \ell$ there exists $1 \leq j \leq k$ such that $q_i = p_j$ and $r_i$ are primes distinct from $p_j$’s such that $s_i = 0$ for each $i$ or $s_i \geq t_i$ for each $i$.

**Proof.** Without loss of generality, put $\omega = p_1^{s_1} p_2^{s_2} \ldots p_m^{s_m} r_1^{\alpha_1} r_2^{\alpha_2} \ldots r_\ell^{\alpha_\ell}$. If $s_i = 0$ for all $1 \leq i \leq m$, then $\gcd(\omega, n) = 1$, thus by Euler’s Theorem,

$$(\omega)^{\phi(n)} \equiv 1 \pmod{n}$$

$$(\omega)^{\phi(n)+1} \equiv \omega \pmod{n}$$

Hence $\omega$ is a trivial semi nilpotent element.

If $s_i \geq t_i$ for $1 \leq i \leq m$, then by Euler’s Theorem,

$$(\omega)^{\phi(p_{m+1} t_{m+1}, p_{m+2} t_{m+2}, \ldots, p_k t_k)} \equiv 1 \pmod{p_{m+1} t_{m+1}, p_{m+2} t_{m+2}, \ldots, p_k t_k}.$$
Thus
\[ p_{m+1}^{t_{m+1}} p_{m+2}^{t_{m+2}} \cdots p_k^{t_k} \equiv (\omega)^{\varphi(p_{m+1}^{t_{m+1}} p_{m+2}^{t_{m+2}} \cdots p_k^{t_k}+1) - \omega. \]
Since \( s_i \geq t_i \) for \( 1 \leq i \leq m \), then \( p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m} | \omega \) which implies
\[ p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m} (\omega)^{\varphi(p_{m+1}^{t_{m+1}} p_{m+2}^{t_{m+2}} \cdots p_k^{t_k}+1) - \omega \cdots (2) \]
From (1) and (2) we obtain:
\[ n \mid (\omega)^{\varphi(p_{m+1}^{t_{m+1}} p_{m+2}^{t_{m+2}} \cdots p_k^{t_k}+1) - \omega, \]
that is
\[ (\omega)^{\varphi(p_{m+1}^{t_{m+1}} p_{m+2}^{t_{m+2}} \cdots p_k^{t_k}+1) - \omega \equiv 0 \mod n). \]
Hence \( \omega \) is a trivial semi nilpotent element.

**Lemma 2.4.** Let \( p_1, p_2, \ldots, p_k \) be distinct primes and \( \lambda = \ell \text{cm} \left( \varphi(p_1), \varphi(p_2), \ldots, \varphi(p_k) \right) \). If \( (a)^{p_i-1} \equiv 1 \mod p_i \) for each \( 1 \leq i \leq k \), then \( a^\lambda \equiv 1 \mod p_1 p_2 \cdots p_k \).

**Proposition 2.5.** Consider \( Z_n \), with the prime factorization of \( n = p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k} \), with at least one of \( t_i > 1 \) and \( 0 \neq \omega \in Z_n \). Then \( \omega \) is a nontrivial semi nilpotent if \( \omega \) has one of the forms:

i) \( \omega = p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k} \ r_1^{a_1} \ r_2^{a_2} \cdots r_m^{a_m} \), such that \( s_1, s_2, \ldots, s_k \) are different from zero.

ii) \( \omega = q_1^{s_1} q_2^{s_2} \cdots q_\ell^{s_\ell} \ r_1^{a_1} \ r_2^{a_2} \cdots r_m^{a_m} \) such that for each \( 1 \leq j \leq \ell \) there exists \( 1 \leq i \leq k \), \( q_j = p_i \) with \( \ell < k \) and \( 0 < s_i < t_i \), for at least one \( i \in \{1, 2, \ldots, \ell \} \).

In both cases \( a_i \geq 0, r_j \) are primes distinct from \( p_1, p_2, \ldots, p_k \).

**Proof.**

i) Since \( \omega \) is divisible by \( p_1, p_2, \ldots, p_k \), so \( \omega \) is a nilpotent element and by Proposition 2.1, \( \omega \) is a nontrivial semi nilpotent element.

ii) Without loss of generality suppose that \( t_1 > 1 \) and \( \omega = p_1^{s_1} p_2^{s_2} \cdots p_\ell^{s_\ell} \ r_1^{a_1} \ r_2^{a_2} \cdots r_m^{a_m} \) for \( 0 < s_1 < t_1 \), \( \ell < k \). By Fermat’s Little Theorem, \( (\omega)^{p_i-1} \equiv 1 \mod p_j \) for each \( \ell + 1 \leq j \leq k \).

Put \( \lambda = \ell \text{cm} \left( \varphi(p_1^{t_1}), \varphi(p_2^{t_2}), \ldots, \varphi(p_k) \right) \). Then by Lemma 2.4,
\[ \omega^\lambda \equiv 1 \mod p_1 p_2 \cdots p_k \]
Hence \( \omega^{\lambda+1} \equiv \omega \mod p_1 p_2 \cdots p_k \).

Put \( \omega^\lambda = \omega \mod p_1 p_2 \cdots p_k \).

\( \omega^{\lambda+1} = \omega \)
consequently \( p_1 p_2 \cdots p_k \mid \omega^{\lambda+1} - \omega \), so \( \omega^{\lambda+1} - \omega \) is a nilpotent element of \( Z_n \). It remains to show that \( \omega^{\lambda+1} = \omega \equiv 0 \mod n \).

Now \( \omega^{\lambda+1} - \omega = (\omega^\lambda - 1) \), then from the fact that \( p_1^{s_1} \mid \omega \), \( p_1^{t_1} \mid \omega \) and \( p_1 \nmid \omega^{\lambda} - 1 \), we deduce that \( p_1^{t_1} \mid \omega^{\lambda+1} - \omega \). So \( \omega^{\lambda+1} - \omega \equiv 0 \mod n \). Therefore \( \omega \) is a nontrivial semi nilpotent element of \( Z_n \).
Proposition 2.6. Every nontrivial nilpotent element in a ring $\mathcal{R}$ is a nontrivial semi nilpotent.

**Proof.** Clearly every nilpotent element is semi nilpotent. Let $0 \neq \omega$ be a nilpotent element of $\mathcal{R}$ and $n$ be the least positive integer such that $\omega^n = 0$. If $\omega^k - \omega = 0$ for some $k$, $1 < k < n$, then $\omega^k = \omega$. Now $0 = \omega^n = \omega^k \omega^{n-k} = \omega^{n-k+1}$ but $n-k+1 < n$, contradiction. Hence $\omega$ is a nontrivial semi nilpotent.

Theorem 2.7. Every element in $\mathbb{Z}_2 G \setminus \{0, 1\}$, where $G$ is a cyclic group of order $2^n$ and $n$ is an even integer is a nontrivial semi nilpotent element.

**Proof.** Let $\omega \in \mathbb{Z}_2 G \setminus \{0, 1\}$. Then $\omega$ is of the form $\omega = a_0 + a_1 g + \cdots + a_{2^n-1} g^{2^n-1}$ such that there is $a_i \neq 0$ for some $i > 0$. If $\omega$ has an even number of nonzero terms say $2\ell$ that is $\omega = a_{i_1} g^{i_1} + a_{i_2} g^{i_2} + \cdots + a_{i_{2\ell}} g^{i_{2\ell}}$, then

$$\omega^{2^n} = \frac{1 + 1 + \cdots + 1}{2\ell-\text{times}} = 0,$$

which means that $\omega$ is a nontrivial nilpotent. Hence by Proposition 2.6, $\omega$ is a nontrivial semi nilpotent. If $\omega$ has an odd number of nonzero terms say $2\ell + 1$, that is $\omega = a_{i_1} g^{i_1} + a_{i_2} g^{i_2} + \cdots + a_{i_{2\ell+1}} g^{i_{2\ell+1}}$, thus

$$\omega^{2^n} = \frac{1 + 1 + \cdots + 1}{(2\ell+1)-\text{times}} = 1,$$

Hence $\omega^{2^n} - \omega = 1 - \omega = 1 + \omega$, so $\omega$ is a nontrivial semi nilpotent element.

In what follows we answer the following open problem given by Vasantha: Let $\mathbb{Z}_p$ be the prime field of characteristic $p$ ($p > 2$) and $G = \langle g : g^q = 1 \rangle$ be a cyclic group of order $q$.

a. If $(p,q) = 1$, can the group ring $\mathbb{Z}_p G$ have nontrivial semi nilpotent elements?

b. If $p | q$, can the group ring $\mathbb{Z}_p G$ have nontrivial semi nilpotent elements?

Now, we need the following Lemma’s.

Lemma 2.8. In $\mathbb{Z}_p G$, $p$ prime, and $G = \langle g : g^q = 1 \rangle$ is a cyclic group of order $q$, we have:

$$(a_0 + a_1 g + a_2 g^2 + \cdots + a_{q-1} g^{q-1})^p = a_0^p + (a_1 g)^p + \cdots + (a_{q-1} g^{q-1})^p$$

Moreover by induction one can show that

$$(a_0 + a_1 g + a_2 g^2 + \cdots + a_{q-1} g^{q-1})^{p^k} = a_0^{p^k} + (a_1 g)^{p^k} + \cdots + (a_{q-1} g^{q-1})^{p^k}$$

Lemma 2.9. For each $a \in \mathbb{Z}_p$ we have $(a)^{p^k} \equiv a \pmod{p}$, for any $k \in \mathbb{N}$.

Theorem 2.10. Let $p_1, p_2, \ldots, p_r$ and $p$ be distinct primes. If $G$ is a cyclic group of order $m$ with the prime factorization of $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then every element in the group ring $\mathbb{Z}_p G$, is a trivial semi nilpotent element.

**Proof.** Suppose $\omega = a_0 + a_1 g + a_2 g^2 + \cdots + a_{m-1} g^{m-1} \in \mathbb{Z}_p G$. 

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We will show that \((\omega)^{p(p_1 k_1 p_2 k_2 \cdots p_r k_r)} = \omega\).

By Lemma 2.8, we obtain:

\[
(\omega)^{p(p_1 k_1 p_2 k_2 \cdots p_r k_r)} = (a_0 + a_1 g + a_2 g^2 + \cdots + a_{m-1} g^{m-1})^{p(p_1 k_1 p_2 k_2 \cdots p_r k_r)} \\
= (a_0 + a_1 g + \cdots + a_{m-1} g^{m-1})^{p(p_1 k_1 - p_1 k_1^{-1})(p_2 k_2 - p_2 k_2^{-1}) \cdots (p_r k_r - p_r k_r^{-1})} \\
= (a_0)^{p(p_1 k_1 - p_1 k_1^{-1})(p_2 k_2 - p_2 k_2^{-1}) \cdots (p_r k_r - p_r k_r^{-1})} \\
+ (a_1 g)^{p(p_1 k_1 - p_1 k_1^{-1})(p_2 k_2 - p_2 k_2^{-1}) \cdots (p_r k_r - p_r k_r^{-1})} \\
+ \cdots + (a_{m-1} g^{m-1})^{p(p_1 k_1 - p_1 k_1^{-1})(p_2 k_2 - p_2 k_2^{-1}) \cdots (p_r k_r - p_r k_r^{-1})} \\

\ldots (*)
\]

By Euler’s Theorem, \(p^{\phi(p_1 k_1 p_2 k_2 \cdots p_r k_r)} \equiv 1 \pmod{p_1 k_1 p_2 k_2 \cdots p_r k_r}\), so \(p(p_1 k_1 - p_1 k_1^{-1})(p_2 k_2 - p_2 k_2^{-1}) \cdots (p_r k_r - p_r k_r^{-1}) \equiv 1 \pmod{p_1 k_1 p_2 k_2 \cdots p_r k_r}\), and by Lemma 2.9, then:

\[
(\omega)^{p^{\phi(m)}} = a_0 + a_1 g + a_2 g^2 + \cdots + a_{m-1} g^{m-1} = \omega.
\]

Consequently, \((\omega)^{p(p_1 k_1 - p_1 k_1^{-1})(p_2 k_2 - p_2 k_2^{-1}) \cdots (p_r k_r - p_r k_r^{-1})} - \omega = 0\).

Therefore, \(\omega\) is a trivial semi nilpotent element.

The following corollary is a direct consequence of Theorem 2.10.

**Corollary 2.11.** The group ring \(\mathbb{Z}_p G\), where \(p\) prime and \(G\) is a cyclic group of order \(q\) such that \(p \nmid q\), has no nontrivial semi nilpotent element.

**Theorem 2.12.** Let \(\mathbb{Z}_p\) be the prime field of characteristic \(p\) \((p \geq 2)\), and \(G\) be a cyclic group of order \(q\). If \(p \mid q\), then the group ring \(\mathbb{Z}_p G\) has nontrivial semi nilpotent elements.

**Proof.** Since \(p \mid q\) then \(q = pm\) for some \(m \in \mathbb{N}\). Now let

\[
\omega = 1 + g^m + g^{2m} + \cdots + g^{(p-1)m} \in \mathbb{Z}_p G.
\]

\[
(\omega)^p = (1 + g^m + g^{2m} + \cdots + g^{(p-1)m})^p
\]

\[
= 1 + (g^m)^p + (g^{2m})^p + \cdots + (g^{(p-1)m})^p \quad \text{(Lemma 2.8)}
\]

\[
= 1 + g^q + g^{2q} + \cdots + g^{(p-1)q} = \frac{1 + 1 + \cdots + 1}{p\text{-times}}
\]

\[
= p (1) = 0.
\]

So \(\omega\) is a nontrivial nilpotent element and by Proposition 2.6, \(\omega\) is a nontrivial semi nilpotent element. Hence the group ring \(\mathbb{Z}_p G\) has nontrivial semi nilpotent elements.
3. Conclusion

In this work the form of trivial and nontrivial semi nilpotent elements obtained in $\mathbb{Z}_n$. It is shown that every element except 0 and 1 in the group ring $\mathbb{Z}_nG$ where $G$ is a cyclic group of order $2^n$, for even $n$, is a nontrivial semi nilpotent element. In addition we have got the answer of the following open problem given by Vasantha: Let $\mathbb{Z}_p$ be the prime field of characteristic $p$ ($p > 2$) and $G = \langle g: g^q = 1 \rangle$ be a cyclic group of order $q$.

a. If $(p,q) = 1$, can the group ring $\mathbb{Z}_pG$ have nontrivial semi nilpotent elements?

b. If $p|q$, can the group ring $\mathbb{Z}_pG$ have nontrivial semi nilpotent elements?

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